CS275 RECITATION EXERCISES

For each question, read **each word** with the greatest care and **without hurrying**. If you have doubts about what is asked, **go back** to the wording of the question until the meaning of the question is clear. Then try to find an answer.

Be careful with notations and words:

$f : R \longrightarrow \mathbb{N}$ reads

"f is a function mapping R into $\mathbb{N}$" or
"f is a function from R to $\mathbb{N}$" or
"f maps R into $\mathbb{N}$."

$f : x \in R \longrightarrow \lfloor x \rfloor \in \mathbb{Z}$ reads

"f maps any real number x to the integer "floor x"" or
"f is the function that associates "floor x" to a real number x. (in addition, its co-domain is $\mathbb{Z}$)."

$[a, b[= [a, b) = \{x \in R \mid a \leq x < b\}$ reads

"the interval a, b, closed on one side and open on the other."

Exercise 1. Let f be the function.

$$f : x \in]0, +\infty[\longrightarrow 2\log(x) \in R$$

a) What are the domain and co-domain of f?
Solution: Domain: $]0, +\infty[$. Co-domain: R.

b) Write in mathematical notation the range of f.
Solution:

$$f(]0, +\infty[) = \{f(x) \mid x \in R\} = \{y \in R \mid \exists x \in R, f(x) = y\}$$

c) Find three elements in the range of f.
Solution:

$$\begin{aligned} 0 &= f(1) \\ -2 &= f(1/e) \\ 2 &= f(e). \end{aligned}$$

d) What is the image of 10 by f?
Solution:

$$f(10) = 2\log(10).$$

e) Find a pre-image of 10 by f.
Solution:

$$\begin{aligned} &f(x) = 10 \iff \\ &2\log(x) = 10 \iff \\ &\log(x) = 5 \iff \\ &x = e^5. \end{aligned}$$

Exercise 2. Let g be the function.

$$g : x \in \mathbb{N} \longrightarrow \sqrt{x} \in R$$

a) What are the domain and co-domain of g?
Solution: Domain: $\mathbb{N}$. Co-domain: R.

b) Write in mathematical notation the range of g.
Solution:

$$g(\mathbb{N}) = \{g(x) \mid x \in \mathbb{N}\} = \{y \in R \mid \exists x \in \mathbb{N},\ g(x) = y\}$$

c) Find three elements in the range of g.
Solution:

$$\begin{aligned} 0 &= g(0) \\ 1 &= g(1) \\ \sqrt{2} &= g(2). \end{aligned}$$

d) What is the image of 10 by g?
Solution:

$$g(10) = \sqrt{10}.$$

e) Find a the pre-image of 10 by g.
Solution:

$$\begin{aligned} &g(x) = 10 && \Longleftrightarrow \\ &x \in \mathbb{N} \text{ and } \sqrt{x} = 10 && \Longleftrightarrow \\ &x \in \mathbb{N} \text{ and } x = 100 && \Longleftrightarrow \\ &x = 100. \end{aligned}$$

f) Find a pre-image of $1/2$ by g.
Solution:

$$\begin{aligned} &g(x) = 1/4 && \Longleftrightarrow \\ &x \in \mathbb{N} \text{ and } \sqrt{x} = 1/2 && \Longleftrightarrow \\ &x \in \mathbb{N} \text{ and } x = 1/4. \end{aligned}$$

$1/2$ belongs to the co-domain of g, but it has no pre-image by g, since $(1/2)^2 = 1/4$ does not belong to the domain of g.

Exercise 3. Let h be the function.

$$h : (n, x) \in \mathbb{N} \times R \longrightarrow 1 + x + \ldots + x^n \in R$$

a) What are the domain and co-domain of h?
Solution: Domain: $\mathbb{N} \times R$. Co-domain: R.

b) Write in mathematical notation the range of h.
Solution:

$$h(\mathbb{N} \times R) = \{h(n, x) \mid n \in \mathbb{N},\ x \in R\} = \{y \in R \mid \exists x \in R,\ \exists n \in \mathbb{N},\ h(n, x) = y\}$$

c) Find three elements in the range of h.
Solution:

$$\begin{aligned} 1 &= h(0,1) = h(0,100) \\ 11 &= h(1,10) \\ 1 + \sqrt{2} &= h\left(1, \sqrt{2}\right). \end{aligned}$$

d) What is the image of $(2, 10)$ by h?
Solution:

$$h(2,10) = 1 + 10 + 100 = 111 \quad .$$

e) Find a pre-image of 10 by h.
Solution:

$$h(1,9) = 10.$$

Exercise 4. Let r be the function.

$$r \left\{ \begin{array}{ccc} \mathbb{Z} \times R & \longrightarrow & R \\ (m, x) & \longrightarrow & 1 + x + \ldots + x^m \end{array} \right.$$

Compare this function to the previous function.

Exercise 5. Let s be the function.

$$s \, : \, u \in \{0..99\} \longrightarrow \left(\left\lfloor \frac{u}{10} \right\rfloor \, , \, u - 10 \cdot \left\lfloor \frac{u}{10} \right\rfloor \right) \in \mathbb{N} \times \mathbb{N}$$

a) What are the domain and co-domain of s?
Solution: Domain: $\{0..99\}$. Co-domain: $\mathbb{N} \times \mathbb{N}$.

b) Write in mathematical notation the range of s.
Solution:

$$s(\{0..99\}) = \{s(n) \mid n \in \{0..99\}\} = \{(m,n) \in \mathbb{N} \times \mathbb{N} \mid \exists u \in \{0..99\}, \, s(u) = (m,n)\}.$$

c) Find three elements in the range of s.
Solution:

$$\begin{aligned} (0,0) &= s(0) \\ (1,2) &= s(12) \\ (9,9) &= s(99). \end{aligned}$$

d) What is the image of 10 by s?
Solution:

$$s(10) = (1,0).$$

e) Find a pre-image of $(7,6)$ by s.
Solution:

$$s(76) = (7,6).$$

Exercise 6. What is wrong in the following function definition?

$$f \, : \, x \in \mathbb{N} \longrightarrow \sqrt{x} \in \mathbb{N}$$

Solution: As it is defined, $f(x)$ does not always belong to the co-domain: $f(2) \notin \mathbb{N}$.

Exercise 7. For each of the following functions,

a) Either prove that it is onto or prove that it is not onto.
b) Either prove that it is one-to-one or prove that it is not one-to-one.
c) Either prove that it is a bijection or prove that it is not a bijection.

$$\begin{aligned} f \, : \, x \in R^+ &\longrightarrow \sqrt{x} \in R \\ g \, : \, x \in R^+ &\longrightarrow \sqrt{x} \in R^+ \\ h \, : \, x \in \mathbb{N} &\longrightarrow \sqrt{x} \in R^+ \\ r \, : \, x \in R &\longrightarrow (x, 2x) \in R \times R \\ s \, : \, x \in R &\longrightarrow 2x \in R \\ t \, : \, x \in \mathbb{N} &\longrightarrow 2x \in \mathbb{N} \end{aligned}$$

Solution:

$f \, : \, x \in R^+ \longrightarrow \sqrt{x} \in R$
a) $-1 \in R$ and $\forall x \in R^+$, $\sqrt{x} \neq -1$, so f is not onto.
b) $\forall x \in R^+$, $\forall y \in R^+$, $x \neq y \Longrightarrow \sqrt{x} \neq \sqrt{y}$, so f is one-to-one.
c) f is not a bijection, since it is not onto.

$g \, : \, x \in R^+ \longrightarrow \sqrt{x} \in R^+$
a) $\forall y \in R^+$, $\exists x = y^2 \in R^+$, s.t. $g(x) = y$, so g is onto.
In short, one could say $\forall y \in R^+$, $y^2 \in R^+$ and $g(y^2) = y$.
b) $\forall x \in R^+$, $\forall y \in R^+$, $x \neq y \Longrightarrow \sqrt{x} \neq \sqrt{y}$, so g is one-to-one.
c) g is a bijection, since it is onto and one-to-one.

$h \, : \, x \in \mathbb{N} \longrightarrow \sqrt{x} \in R^+$
a) $\sqrt{1/4} = 1/2 \in R^+$, $\forall x \in \mathbb{N}$, $\sqrt{x} \neq 1/2$, so h is not onto.

b) $\forall x \in \mathbb{N}$, $\forall y \in \mathbb{N}$, $x \neq y \Longrightarrow \sqrt{x} \neq \sqrt{y}$, so h is one-to-one.
c) h is not a bijection, since it is not onto.

$r : x \in R \longrightarrow (x, 2x) \in R \times R$
a) $(0, 1) \in R \times R$, $\forall x \in R$, $(x, 2x) \neq (0, 1)$, so r is not onto.
b) $\forall x \in R$, $\forall y \in R$, $x \neq y \Longrightarrow (x, 2x) \neq (y, 2y)$, so r is one-to-one.
c) r is not a bijection, since it is not onto.

$s : x \in R \longrightarrow 2x \in R$
a) $\forall y \in R$, $\exists x = y/2 \in R$, $s(x) = y$, so s is onto.
b) $\forall x \in R$, $\forall y \in R$, $x \neq y \Longrightarrow 2x \neq 2y$, so s is one-to-one.
c) s is a bijection, since it is onto and one-to-one.

$t : x \in \mathbb{N} \longrightarrow 2x \in \mathbb{N}$
a) $1 \in \mathbb{N}$ and $\forall x \in \mathbb{N}$, $2x \neq 1$, so t is not onto.
b) $\forall x \in \mathbb{N}$, $\forall y \in \mathbb{N}$, $x \neq y \Longrightarrow 2x \neq 2y$, so t is one-to-one.
c) t is not a bijection, since it is not onto.

References