CS275 WEEK 14 RECITATION EXERCISES - SOLUTION

For each question, read **each word** with the greatest care and **without hurrying**. If you have doubts about what is asked, **go back** to the wording of the question until the meaning of the question is clear. Then try to find an answer.

1. Reminders

2. Exercises

Exercise 1. Note that this is part of Ex. 4. a of Homework 9. This will nevertheless be graded; please write the solution clearly and, if possible, varying the wording wrt that of the solution, which will be posted on Monday.
Determine the number of leaves of a complete m-ary tree of depth h, for $m > 1$. Find a proof by induction of your result.

a) Draw complete m-ary trees of depth h, for $m \in \{2, 3\}$ and $h \in \{0, 1, 2\}$.
Solution: Should not be needed.

b) Write an educated guess for the number of leaves of a complete m-ary tree of depth h.
Solution: It seems to be m^h, right?

c) Write down the induction hypothesis.
Solution: Let $P(h)$ be the proposition "all complete m-ary trees of depth h have m^h leaves."

d) Show that a full m-ary tree of depth 0 has 1 leaf.
Solution: The only (up to isomorphism) complete m-ary tree of heigth 0 is $T = (\{0\}, \emptyset)$, which has $1 = m^0$ leaf.

e) Let T be a full m-ary tree of depth $D+1$, for some $D \geq 0$ and T' be T without its leaves and the corresponding edges. Let $u_1, \ldots, u_N$ be the leaves of T and $v_1, \ldots, v_Q$ the leaves of T'. Note that the v_i are also nodes (but not leaves) of T.
- What is the depth of T'? .... D.
- Assuming $P(D)$ is true, how many leaves does T' have? .... m^D.
- How many children does each v_i have in T? .... m.
- How many children, in T, do $v_1, \ldots, v_Q$ have altogether? .... $Q \cdot m = m^D \cdot m = m^{D+1}$.

f) Write a complete proof by induction using the results in **d)** and **e)**.
Solution:
Basis step: $P(0)$ is proven in **d)**.
Induction step: Assume $P(D)$ is true , for some $D \geq 0$.
Let T be a full m-ary tree of depth $D+1$ and T' be T with its leaves and the corresponding edges removed. Let $v_1, \ldots, v_Q$ be the leaves of T'. Note that the v_i are also nodes (but not leaves) of T. It is clear that T' is complete and has depth D. By the induction hypothesis, T' has m^D leaves. Since each v_i has m children in T and the children are all distinct, the v_i have a total of $m \cdot m^D = m^{D+1}$ children. Since the children of the v_i are the leaves of T, T has m^{D+1} leaves.

Exercise 2. Somewhat like Exercises 3-4 p. 642 of [1]. Answer these questions about the trees illustrated in Figure 2.1.

a) Which vertex is the root?
Solution: Should not be needed.

b) Which vertices are leaves?
Solution: Should not be needed.

c) Which vertices are in internal?
Solution: Should not be needed.

d) Which vertex is the parent of h?
Solution: c.

e) Which vertices are children of l?
Solution: 1:r, s, 2:q, r, s, 3:q, r, s.

f) Which vertices are siblings of n?
Solution: 1:m, o, 2:o, p, 3:o, p.

g) Which vertices are ancestors of p?
Solution: 1:j, d, a, 2:h, c, a, 3:h, c, a.

h) Which vertices are descendants of c?
Solution: 1:g, m, n, o, h, 2:h, n, o, p, i, j, 3:h, n, o, p, i, j.

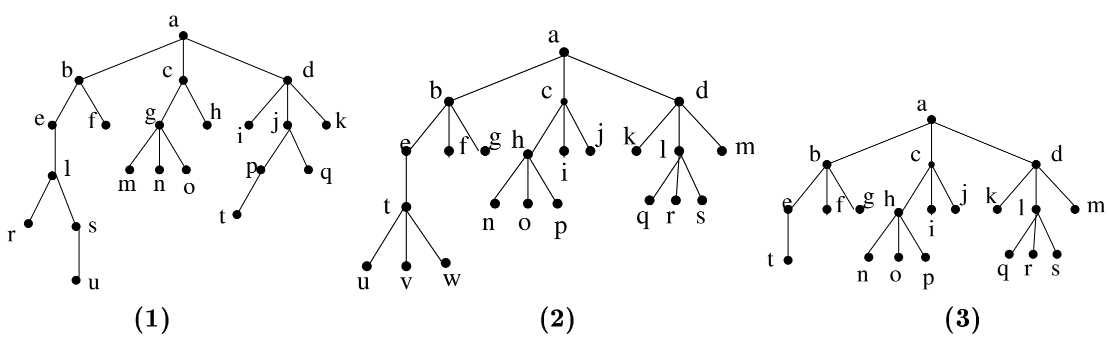

FIGURE 2.1. Trees for Exercises 2 and 3.

Exercise 3. Somewhat like Exercises 5-6 p. 642 of [1]. Answer these questions about the trees in Figure 2.1.

a) At what depth are the leaves? .... Easy.
b) Which of these trees are balanced? .... (3) only.
c) What is the depth/height of these trees? .... 5, 4 and 3.
d) For what m are these trees m-ary trees? .... Any $m \geq 3$.
e) Which of these trees are full m-ary trees for some (which) m? .... (2) is a full ternary tree.

Exercise 4. Solve exercises 11, 12 and 13 p. 642 of [1].

a) How many nonisomorphic unrooted trees are there with 3, 4 and 5 vertices?
b) How many nonisomorphic rooted trees are there with 3, 4 and 5 vertices (using isomorphisms for directed graphs)?

Solution:

a) 1, 2 and 3.
b) 2, 4 and 9.

See Figure 4.1

References

[1] K. H. Rosen. *Discrete Mathematics and Its Applications*. Mc Graw Hill, 5 edition, 2003.

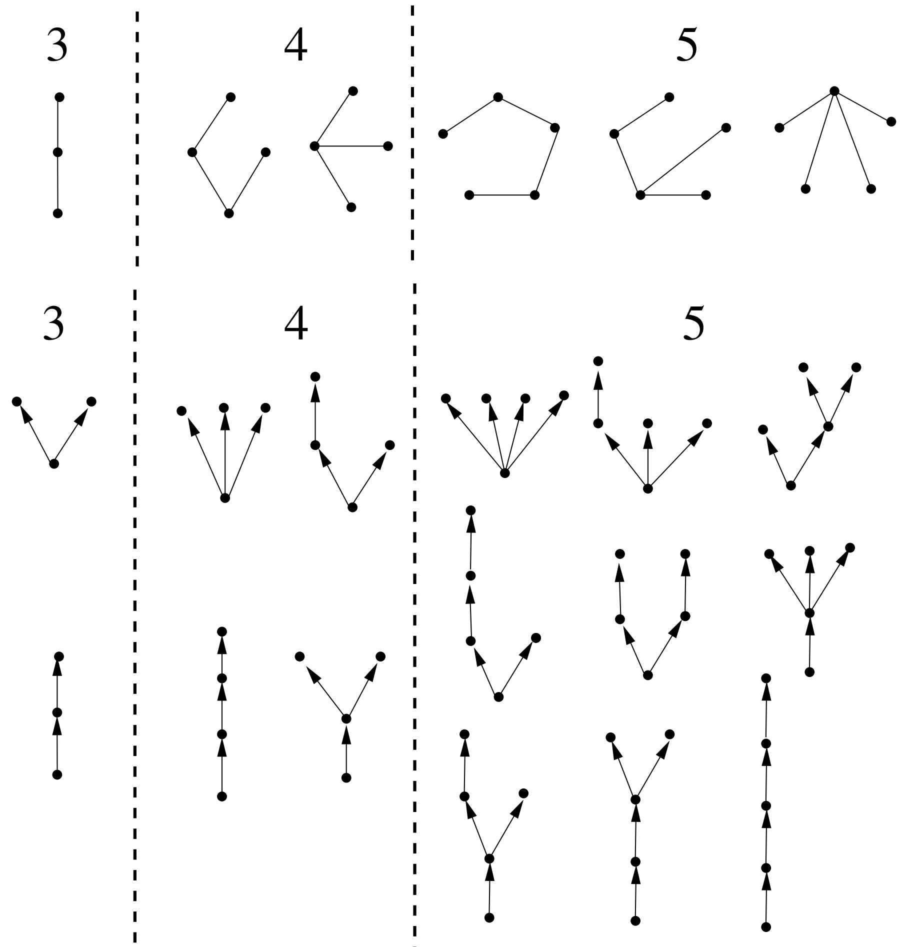

FIGURE 4.1. Non isomorphic unrooted **(top)** and rooted **(bottom)** trees of size 3, 4 and 5, for Exercise 4.