# WEEK 10 : CS275 RECITATION EXERCISES - SOLUTION

For each question, read **each word** with the greatest care and **without hurrying**. If you have doubts about what is asked, **go back** to the wording of the question until the meaning of the question is clear. Then try to find an answer.

**Exercise 1.** Use inverters, AND and OR gates to build circuits with these outputs:

**a)** $(x+y)(x+z)$
**b)** $x+yz$
**c)** $xyz+\bar{x}\bar{y}\bar{z}$
**d)** $(x+\bar{y})(y+\bar{z})(z+\bar{x})$

**Exercise 2.** Build circuits using only NAND gates, to produce these outputs:

**a)** $\bar{x}$ ........ $= x|x$
**b)** $xy$ ........ $= \overline{\overline{xy}} = \overline{x|y} = (x|y)\,|\,(x|y)$
**c)** $x+y$ ........ $= \overline{\bar{x}\bar{y}} = \bar{x}|\bar{y} = (x|x)\,|\,(y|y)$
**d)** $x(y+\bar{z})$ ........ $(x|\,((y|y)\,|z))\,|\,(x|\,((y|y)\,|z))$

**Exercise 3.** Determine whether the XOR, NAND and NOR operators ($\oplus$, $|$ and $\downarrow$) are associative, i.e. if, for all $x$, $y$ and $z$ in $\{0,1\}$, one has $x\oplus(y\oplus z)=(x\oplus y)\oplus z$, $x|\,(y|z)=(x|y)\,|z$ and $x\downarrow(y\downarrow z)=(x\downarrow y)\downarrow z$. Does it make sense to write, then $x\oplus y\oplus z$, $x|y|z$ and $x\downarrow y\downarrow z$ (or to build three-input XOR, NAND [1] and NOR gates)?

**Solution:**

**a)** $\forall x,\ y,\ x,\ x\oplus(y\oplus z)=(x\oplus y)\oplus z$. Just do the truth table. Associative.
**b)** $1|\,(1|0)=1|1=0\neq 1=(1|1)\,|0=0|0$. Not associative.
**c)** $1\downarrow(1\downarrow 0)=1\downarrow 0=0\neq 1=(1\downarrow 1)\downarrow 0=0\downarrow 0=1$. Not associative.

Notation: $x\oplus y\oplus z$ can always be read correctly as either $x\oplus(y\oplus z)$ or $(x\oplus y)\oplus z$, since the value is the same. I am not aware of a three- or more- input XOR chip, though. Writing $x|y|z$ or $x\downarrow y\downarrow z$ would be trickier since $x|y|z$ could mean $x|\,(y|z)$, $(x|y)\,|z$ or $\overline{xyz}$, this last meaning being the one used for three- and more- input chips.

**Exercise 4.** Solve Exercise 10 p. 718 of [1]. Construct a circuit for a half subtractor using AND gates, OR gates and inverters. A **half subtractor** has two bits as input and produces as output a difference bit and a borrow.

Hint: You may want to build the truth table of $x-y$ and the truth table of the borrow bit.

**Solution:** The truth table is

| $x$ | $y$ | $x-y$ | $b$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 0 |

So you could use e.g. $x-y=x\oplus y=\bar{x}y+x\bar{y}$ and $b=\bar{x}y$.

[1]Note: there exist 3-input NAND and NOR gates (e.g. 74LS10 and 74LS27), which output $\overline{xyz}$ and $\overline{x+y+z}$.

**Exercise 5.** Solve Exercise 8 p. 423 of [1]. A model for the number of lobsters caught per year is based on the assumption that the number of lobsters caught in a year is the average of the number caught in the two previous years.

**a)** Find a recurrence relation for $\{L_n\}$, where $L_n$ is the number of lobsters caught in year $n$, under the assumption for this model.
**Solution:**

$$L_n = \frac{1}{2}\left(L_{n-1} + L_{n-2}\right) = \frac{1}{2}L_{n-1} + \frac{1}{2}L_{n-2}. \tag{5.1}$$

**b)** Find $L_n$ if 100,000 lobsters are caught in year 1 and 300,000 were caught in year 2.
**Solution:**
Straight out of the blackboard of Thursday´s class: Let $r_1 = \frac{\frac{1}{2}+\sqrt{\left(\frac{1}{2}\right)^2+4\frac{1}{2}}}{2} = 1$ and $r_2 = \frac{\frac{1}{2}-\sqrt{\left(\frac{1}{2}\right)^2+4\frac{1}{2}}}{2} = -\frac{1}{2}$ be the solutions to $r^2 - \frac{1}{2}r - \frac{1}{2} = 0$ - this is called the "**characteristic equation**" of the relation Eq. 5.1, I will come back to this briefly next Thursday.
Since $r_1 \neq r_2$, all the solutions to Eq. 5.1 take the form $L_n = \alpha_1 r_1^n + \alpha_2 r_2^n$, for some $\alpha_1$ and $\alpha_2$. The only parameters $\alpha_1$, $\alpha_2$ for which the initial conditions $L_0 = 1e5$ and $L_1 = 3e5$ hold are: $\alpha_1 = \frac{r_2 L_0 - L_1}{r_2 - r_1} = \frac{7}{3}e5$ and $\alpha_2 = \frac{-r_1 L_0 + L_1}{r_2 - r_1} = -\frac{4}{3}e5$. Altogether, one has:

$$L_n = 1e5\left(\frac{7}{3} - \frac{4}{3}\left(-\frac{1}{2}\right)^n\right).$$

**Exercise 6.** Find the solutions to the homogeneous recurrence relations and compute the first few terms of the sequence

**a)** $x_n = 2x_{n-1} - x_{n-2}$ with initial conditions $x_0 = 2$ and $x_1 = -4$
**Solution:** Since $r_1 = r_2 = 1 = r$, the solutions take the form $x_n = \alpha r^n + \beta n r^n$, where $\alpha = x_0 = 2$ and $\beta = \frac{x_1}{r} - x_0 = -6$. Altogether, one has

$$x_n = 2 - 6n.$$

The first values of the sequence are : 2, -4, -10, -16 ...

**b)** $x_n = x_{n-1} + \frac{3}{4}x_{n-2}$ with initial conditions $x_0 = -2$ and $x_1 = 1$
**Solution:** The solutions $r_1 = \frac{3}{2}$ and $r_2 = -\frac{1}{2}$ to the characteristic equation are distinct, so that the solutions of the recurrence relation take the form $x_n = \alpha_1 r_1^n + \alpha_2 r_2^n$, where $\alpha_1 = \frac{x_0 r_2 - x_1}{r_2 - r_1} = 0$ and $\alpha_2 = \frac{-x_0 r_1 + x_1}{r_2 - r_1} = -2$. Altogether, one has

$$x_n = -2\left(-\frac{1}{2}\right)^n.$$

The first few values of the sequence are : -2, 1, -0.5, 0.25, -0.125, 0.0625, -0.0312, 0.0156, -0.00781, 0.00391, -0.00195, 0.000977 ....

**c)** $x_n = x_{n-1} + \frac{3}{4}x_{n-2}$ with initial conditions $x_0 = -\frac{3}{2}$ and $x_1 = 1$
**Solution:** Same as above, but $\alpha_1 = \frac{1}{8}$ and $\alpha_2 = -\frac{13}{8}$, so that

$$x_n = \frac{1}{8}\left(\frac{3}{2}\right)^n - \frac{13}{8}\left(-\frac{1}{2}\right)^n.$$

The first few values of the sequence are : -1.5, 1, -0.125, 0.625, 0.531, 1, 1.4, 2.15, 3.2, 4.81, 7.21, 10.8, 16.2, 24.3, 36.5, 54.7, 82.1, 123, 185, 277...

Note that the same recurrence relation may yield a sequence that tends to zero (**b)** above) or that tends to infinity (**a)**).

## References

[1] K. H. Rosen. *Discrete Mathematics and Its Applications*. Mc Graw Hill, 5 edition, 2003.