BASIC GRAPH DEFINITIONS

Abstract. Graph definitions given in CS275. As I said, definitions in graph theory vary from source to source. What matters is that you grasp the notions that are represented by these definitions.

1. Graph types

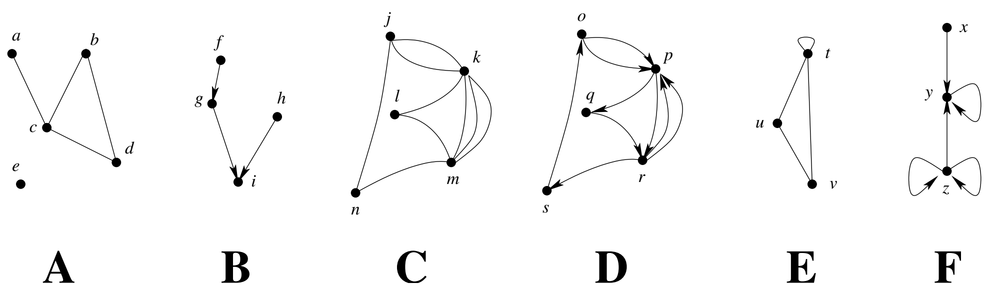

Figure 1.1. Different types of graphs. Simple, directed, directed multi-graph, multi-graph, (undirected) pseudo-graph, (directed) pseudo-graph.

Definition 1.1. Simple graph. A “simple graph” is a pair (V, E), where V is a set of “vertices” and V is a set of “edges”. Each element of V is a (unordered) pair $\{u, v\}$, for some u, $v \in V$. More formally,

$$(V, E) \text{ is a } \textbf{simple graph} \equiv \forall e \in E,\ \exists u, v \in V,\ u \neq v \text{ and } e = \{u, v\}.$$

Definition 1.2. Directed graph. A “directed graph” is a pair (V, E), where V is a set of vertices and V is a set of edges. Each element of V is an ordered pair (u, v), for some u, $v \in V$. More formally,

$$(V, E) \text{ is a } \textbf{directed graph} \equiv \forall e \in E,\ \exists u, v \in V\ u \neq v \text{ and } e = (u, v).$$

Definition 1.3. Weighed graph - multi-graph - multiplicity: A “weighed graph” is a triplet (V, E, w) where, in addition to V and E, one has a “weight function” $w : E \longrightarrow \mathbb{N}$. Given an edge $e \in E$, $w(e)$ will be called its multiplicity.

Some remarks:

- **a)** This definition is used for both undirected and directed graphs.
- **b)** In an undirected graph with loops (see Sec. 1.4 below), it is convenient to define $w(\{t\}) = 2$ for a simple loop, such as $\{t\}$ in Figure 1.1 E.
- **c)** In the undirected case (respectively, directed), it is convenient to define $w(\{\alpha, \beta\}) = 0$ (resp. $w((\alpha, \beta)) = 0$) for all pairs $\alpha, \beta \in V$ s.t. $\{\alpha, \beta\} \notin E$ (resp. $(\alpha, \beta) \notin V$).
- **d)** This is way of representing edge multiplicity if different from that of [1].

Definition 1.4. Graphs with loops: pseudo-graph In pseudo-graphs, one relaxes the condition $u \neq v$ in Definitions 1.1 and 1.2 above. One thus allows edges of the form (in an undirected pseudo-graph) $\{t\}$ or (in a directed pseudo-graph) (x, x). These edges are called “loops”.

Definition 1.5. Undirected graph A simple graph or pseudo-graph in which the edges are sets $\{u, v\}$ rather than ordered pairs (u, v).

See the examples of graph definitions in Figure 1.1 and Table 1.

Vertices V and edges E.

A: $V = \{a, b, c, d, e\}$, $E = \{\{a, c\}, \{b, c\}, \{b, d\}, \{c, d\}\}$
B: $V = \{f, g, h, i\}$, $E = \{(f, g), (g, i), (h, i)\}$
C: $V = \{j, k, l, m, n\}$, $E = \{\{j, k\}, \{j, n\}, \{k, l\}, \{k, m\}, \{l, m\}, \{m, n\}\}$
D: $V = \{o, p, q, r, s\}$, $E = \{(o, p), (p, q), (p, r), (q, r), (r, p), (r, s), (s, o)\}$
E: $V = \{t, u, v\}$, $E = \{\{t\}, \{t, u\}, \{t, v\}, \{u, v\}\}$
F: $V = \{x, y, z\}$, $E = \{(x, y), (y, y), (z, y), (z, z)\}$

Multiplicities - weight function w.

A,B,E: Not needed.
C: $w(\{k, m\}) = 3$, $w(\{j, k\}) = 2$, $w(\{j, n\}) = w(\{k, l\}) = w(\{m, n\}) = 1$ and $w(\{\alpha, \beta\}) = 0$ for all other pairs $\alpha, \beta \in V$.
D: $w(o, p) = 2^a$, $w(p, q) = 1$, $w(p, r) = 1$, $w(q, r) = 1$, $w(r, p) = 2$, $w(r, s) = 1$, $w(s, o) = 1$. $w(\alpha, \beta) = 0$ for all other pairs $\alpha, \beta \in V$.
F: $w(x, y) = w(y, y) = w(z, y) = 1$, $w(z, z) = 2$.

[a] It would be more rigorous to write $w((o, p)) = 2$, but let´s spare a pair of parentheses.

TABLE 1. Mathematical objects that describe the graphs of Figure 1.1.

2. Extra vocabulary

Definition 2.1. Adjacent vertices In an undirected graph (V, E)or multi-graph (V, E, w) or pseudo-graph, a vertex u is "adjacent" to a vertex v iff $\{u, v\}$ belongs to V.

In a directed graph (V, E) or multi-graph (V, E, w) or directed pseudo-graph, a vertex u is adjacent to a vertex v iff (u, v) or (v, u) belongs to V.
In an edge (u, v), u is called the "**start point**" or "**initial vertex**" and v is the "**endpoint**" or "**terminal vertex**."

Definition 2.2. Incidence of an edge and a vertex In an undirected graph (V, E)or multi-graph (V, E, w) or pseudo-graph, an edge $e \in E$ is "incident" to a vertex $u \in V$ iff $u \in e$.

In a directed graph (V, E) or multi-graph (V, E, w) or directed pseudo-graph, an edge $e \in E$ is incident to a vertex $u \in V$ iff $e = (u, v)$ or $e = (v, u)$ for some $v \in V$.

Definition 2.3. Degree of a vertex of a simple graph (undirected graph) The "degree" of a vertex v in a simple graph (V, E) is the number of vertices that are adjacent to it.

The degree of a vertex v in an undirected pseudo-graph (V, E) is the number of vertices that are adjacent to it, where loops are counted twice.
The degree can thus be defined mathematically by

$$\deg(v) = \begin{cases} |\{u \in V \mid v \neq u \,\text{and}\, \{u, v\} \in E\}| & \text{if } \{u\} \notin V, \\ |\{u \in V \mid v \neq u \,\text{and}\, \{u, v\} \in E\}| + 2 & \text{if } \{u\} \in V. \end{cases}$$

In a simple, multi- or pseudo-graph (V, E, w), the degree is the sum of the multiplicites of edges incident to it, again w/ loops counted twice[1].

$$\deg(v) = \sum_{u \in V} w(\{u, v\}).$$

In a directed graph, these definitions become.

Definition 2.4. Degree, in- and out-degree of a vertex (directed graph) The "in-degree" of a vertex v in a directed graph (V, E) is the number of edges that end at v. It can thus be defined as

$$\deg^-(v) = |\{u \in V \mid (u, v) \in E\}|.$$

[1] Loops are counted twice in this expresssion because $w(\{v\})$ was taken to be 2 in Section 1.3.

Degree, in-degree, out-degree.

Undirected	A					C					E		
Graphs	a	b	c	d	e	j	k	l	m	n	t	u	v
deg	1	2	3	2	0	3	6	3	5	2	4	2	2

Directed	B				D					F		
Graphs	f	g	h	i	o	p	q	r	s	x	y	z
$\deg^-$	0	1	0	2	1	4	1	2	1	0	1	2
$\deg^+$	1	1	1	0	2	2	1	3	1	1	3	3
deg	1	2	1	2	3	6	2	5	2	1	4	5

TABLE 2. Properties of the graphs of Figure 1.1.

In the case of a directed multi-graph (V, E, w), one defines

$$\deg^-(v) = \sum_{u \in V} w(u, v).$$

The "out-degree" of a vertex v in a directed graph (V, E) is the number of edges that start from v. It can thus be defined as

$$\deg^+(v) = |\{u \in V \mid (v, u) \in E\}|.$$

In the case of a directed multi-graph (V, E, w), one defines

$$\deg^+(v) = \sum_{u \in V} w(v, u).$$

Note that these definitions work for graphs with loops (pseudo-graphs) too.

The "degree" of a vertex v in a directed graph (V, E) is the number of edges that start from or end at v. It can thus be defined as

$$\deg(v) = \deg^+(v) + \deg^-(v).$$

Remarks:

- These definitions are simpler than the corresponding definitions for undirected graphs.
- The degree of a vertex of a directed graphs is the same degree it has in the underlying undirected graph, see Section 2.5 below.

The degrees of the vertices in Figure 1.1 are listed in Table 2.

Definition 2.5. Underlying undirected graph of a directed graph The "underlying undirected graph" of a directed graph (V, E) is the undirected graph (V, E') obtained by replacing each edge $(u, v) \in E$ by the (undirected) edge $\{u, v\}$. E' is thus defined by

$$E' = \{\{u, v\} \mid (u, v) \in E\}.$$

Definition 2.6. Pendant vertex A vertex is "pendant" if it is adjacent to only one vertex, i.e. if it´s degree is one.

Definition 2.7. Isolated vertex A vertex is "isolated" if it is adjacent to no vertex, i.e. if it´s degree is zero.

REFERENCES

[1] K. H. Rosen. *Discrete Mathematics and Its Applications.* Mc Graw Hill, 5 edition, 2003.

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