CS275 GRADED HOMEWORK 9

GIVE BACK ON TUESDAY NOVEMBER 30TH 2004 AT BEGINNING OF CLASS

For each question, read **each word** with the greatest care and **without hurrying**. If you have doubts about what is asked, **go back** to the wording of the question until the meaning of the question is clear. Then try to find an answer. If you get stuck, **contact** your T.A. or me.

Please write your section number and an estimate of the time you spent on this HW.

1. Reminders

1.1. Graph isomorphisms.

Definition. Isomorphic graphs. Let $G = (V, E)$ and $G' = (V', E')$ be simple graphs. G and G' are said to be isomorphic if and only if and only if there exists a bijection $f : V \longrightarrow V'$ such that

$$\forall u, v \in V, \quad \{u, v\} \in E \iff \{f(u), f(v)\} \in E'.$$

This definition is valid for graphs with loops too. In the case of multi-graphs (as defined in the helper page `http://www.vis.uky.edu/~etienne/cs275/helpers.html#graphs`), we will use the definition:

Definition. Isomorphic weighed/multi- graphs. Let $G = (V, E, w)$ and $G' = (V', E', w)$ be multi-graphs. G and G' are said to be isomorphic if and only if and only if there exists a bijection $f : V \longrightarrow V'$ such that

$$\forall u, v \in V, \ w(\{u, v\}) = w(\{f(u), f(v)\}).$$

Isomorphic directed graphs are likewise defined by replacing unordered pairs by ordered pairs, i.e. by replacing the curly brackets {} with parentheses ().

Definition. Graph Isomorphism. The function f in the definitions above is called a graph isomorphism from G to G' or between G and G'.

Proposition. *If f is a graph isomorphism between two graphs $G = (V, E)$ and $G' = (V', E')$, then*

- **a)** The number of vertices is preserved: $|V| = |V'|$.
- **b)** The number of edges is preserved: $|E| = |E'|$.
- **c)** The degree of vertices is preserved. For all $u \in V$, one has $\deg(u) = \deg(f(u))$. In the case of directed graphs, one also has $\deg^+(u) = \deg^+(f(u))$ and $\deg^-(u) = \deg^-(f(u))$.
- **d)** In weighed graphs (multi-graphs), the multiplicity of vertices is preserved : for all $u, v \in V$, one has $w(\{u, v\}) = w(\{f(u), f(v)\})$.
- **e)** Paths and circuits are preserved: the image of a path (resp. circuit) in G is a path (resp. circuit) in G' with same length.
- **f)** For any number $d \in \mathbb{N}$, the number of vertices of G with degree d is the same as the number of vertices of G' with degree d. That is, in mathematical notation:

  $$\forall d \in \mathbb{N}, |\{v \in V \mid \deg(v) = d\}| = |\{v' \in V' \mid \deg(v') = d\}|.$$

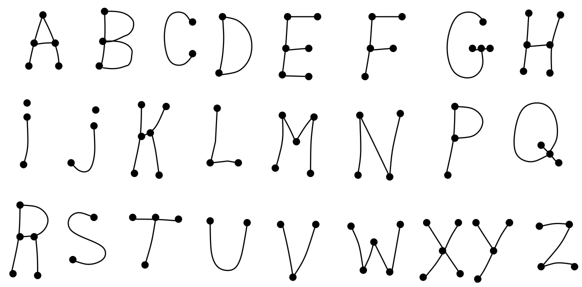
  The same holds for the in- and out- degrees of a directed graph.

1.2. Trees.

Definition. Full m-ary tree. An m-ary tree T if full if and only if each vertex has 0 or m children.

Definition. Balanced rooted tree. A tree T with height (depth) h is balanced if and only if each leaf is at level h or $h - 1$.

Definition. Complete rooted m-ary tree. A rooted m-ary tree T is complete if and only if it is full and all leaves are at the same level.



These drawings of graphs could be represented by the weighed pseudo-graphs:

- A: Vertices: $V = \{1, \ldots, 5\}$, e.g. numbered top to bottom and left to right. Edges: $E = \{\{1,2\}, \{1,3\}, \{2,3\}, \{2,4\}, \{3,5\}\}$. Multiplicity: $w(e) = 1$ for all edge e.
- B : $V = \{1,2,3\}$. $E = \{\{1,2\}, \{2,3\}\}$. $w(e) = 2$ for all edge e.
- C : $V = \{1,2\}$. $E = \{\{1,2\}\}$. $w(e) = 1$ for all edge e.
- D : $V = \{1,2\}$. $E = \{\{1,2\}\}$. $w(e) = 2$ for all edge e.
- etc...

Notice e.g. that C and D differ only by the multiplicity of their edge.

FIGURE 2.1. Graphs for Exercise 2.

2. Exercises

Exercise 1. Let G, G' and G'' be graphs such that G is isomorphic with G' and G' is isomorphic with G''. Let f_1 (resp. f_2) be a graph isomorphism between G and G' (resp. G' and G'').

a) Prove that G is isomorphic with itself.
b) Prove that G' is isomorphic with G, e.g. by showing that f_1^{-1} is a graph isomorphism.
c) Prove that G is isomorphic with G'', e.g. by showing that $f_1 \circ f_2$ is a graph isomorphism.

Exercise 2. Let R be the relation defined on pairs of graphs.

$$R(G, G') \equiv G \,\text{and}\, G' \text{ are isomorphic.}$$

a) Show that R is an equivalence relation (you may want to use results from Ex. 1).
b) Consider the 25 graphs A, B, etc in Figure 2.1. Using the properties listed in the "Reminder" section above, find a sufficient reason for which
  1) A is not isomorphic to B.
  2) A is not isomorphic to F.
  3) H is not isomorphic to E.
c) Let $\mathcal{L}$ be the set of the 25 graphs A, B, etc in Figure 2.1. Write the equivalence classes in $\mathcal{L}$ for the relation R. I.e. find the subsets of $\mathcal{L}$ that contain isomorphic graphs.

Exercise 3. Recall the definitions of Hamilton and Euler circuits, of the complete graph K_n and of the complete bipartite graph $K_{m,n}$.

a) Write the number of Hamilton circuits in K_n as a function of n.
b) For what values of m, n does $K_{m,n}$ have an Euler circuit?
c) For what values of m, n does $K_{m,n}$ have a Hamilton circuit?

Exercise 4. Solve Exercises 28 and 30 of p. 643 [1].

a) How many vertices and how many leaves does a complete m-ary tree of height h have?
b) Show that a full m-ary balanved tree of height h has more than m^{h-1} leaves.

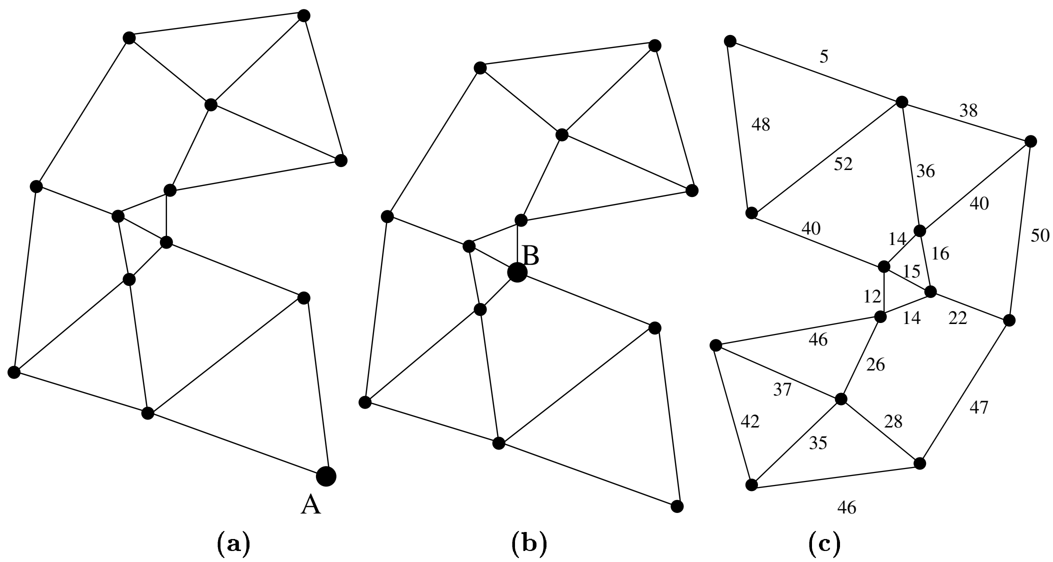


FIGURE 4.1. Graphs for Exercise 5.

Exercise 5. Consider the graphs drawn in Figure 4.1.

a) Draw a spanning tree of the graph in Figure 4.1 (a), rooted in A.
b) Draw a spanning tree of the graph in Figure 4.1 (b), rooted in B.
c) Draw a minimal spanning tree of the graph in Figure 4.1 (c).
d) What is the total length of the edges in this minimal spanning tree?

REFERENCES

[1] K. H. Rosen. *Discrete Mathematics and Its Applications*. Mc Graw Hill, 5 edition, 2003.