CS275 GRADED HOMEWORK

GIVE BACK ON TUESDAY SEP. 21ST 2004 AT BEGINNING OF CLASS

For each question, read **each word** with the greatest care and **without precipitation**. If you have doubts about what is asked, **go back** to the wording of the question until the meaning of the question is clear. Then try to find an answer. If you get stuck, don´t hesitate to **contact** your T.A. or me.

Please see also the example solution of a "prove by induction" exercise, at the end of this document.

Please write your section number on your homework as well as a rough estimate of the time you spent solving it.

Exercise 1. Let A, B and C be the sets defined by

$$\begin{aligned} A &= \{x \in \mathbb{Z} \mid x^2 < 9\} \\ B &= \{x \in \mathbb{N} \mid x^2 \leq 9\} \\ C &= \{x \in \mathbb{N} \mid x \leq 100 \wedge \exists y \in \mathbb{N}, y^2 = x\} \end{aligned}$$

a) Determine the numbers of elements of A, B and C. That is, determine $|A|$, $|B|$ and $|C|$.
Solution:
$|A| = |\{-2, -1, 0, 1, 2\}| = 5.$
$|B| = |\{0, 1, 2, 3\}| = 4.$
$|C| = |\{0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100\}| = 11.$

b) Determine the number of elements in $A \cup B$, $A \cup C$, $A \cup C$, $A \cap B$, $A \cap C$, $B \cap C$.
Solution:
$|A \cup B| = 6.$
$|A \cup C| = 14.$
$|A \cap B| = 3.$
$|A \cap C| = 2.$
$|B \cap C| = 2.$

c) Determine the number of elements in $A \times B$, $A \times C$ and $A \times B \times C$.
Solution:
$|A \times B| = 20.$
$|A \times C| = 55.$
$|A \times B \times C| = 220.$

Exercise 2. Write in English the following statement

$$\forall A \forall B,\ |A \cup B| = |A| + |B|\,.$$

Solution: For all A, for all B, the cardinal of the union of A and B is the sum of the cardinals of A and B.

Prove or disprove this statement.
Solution: This statement is false: Take $A = B = \{1\}$. $|A \cup B| = |A| = 1 \neq |A| + |B| = 2$. So the statement $|A \cup B| = |A| + |B|$ is not true for all A and all B.
Note: You should recall from class that the statement $|A \cup B| = |A| + |B|$ is true **only if** $A \cap B = \emptyset$. See "Sum rule" in the textbook.

Exercise 3. Write in English the following statement

$$\forall A \, \exists B, \; |A \cup B| \neq |A| + |B| \,.$$

Solution: For all A, there exists a B such that the cardinal of the union of A and B is different from the sum of the cardinals of A and B.

Prove or disprove this statement.
Solution: This statement is false. We will prove its negation

$$\exists A \, \forall B, \; |A \cup B| = |A| + |B| \,.$$

Take $A = \emptyset$. For any set B, $A \cup B = B$ and one has $|A \cup B| = |B|$ and $|A| + |B| = 0 + |B| = |B|$ too. So the statement $\forall B$, $|A \cup B| = |A| + |B|$ is true for when $A = \emptyset$.
Note: There were many false proofs of the type: "take $A = \{1, 2\}$ and $B = \{1, 2\}$. Then $|A \cup B| = 2 \neq 2 + 2 = |A| + |B|$." This is a valid proof of the statement

$$\exists A \, \exists B, \; |A \cup B| \neq |A| + |B| \,,$$

which is weaker than the desired statement

$$\forall A \, \exists B, \; |A \cup B| \neq |A| + |B| \,,$$

which is false.

Exercise 4. Write in English the following statement

$$\exists A \, \exists B, \; |A \times B| = |A| \cdot |B| \,.$$

Solution: There exists an A, there exists a B such that the cardinal of the Cartesian product of A and B is equal to the product of the cardinals of A and B.

Prove or disprove this statement.
Solution: This statement is true: Take $A = B = \{1\}$ (or any set, for that matter). One has $|A \times B| = 1 = 1 \cdot 1 = |A| \cdot |B|$.
Note: See "product rule" in the textbook.

Exercise 5. Write, in mathematical notation, the negations of the following statements

a) $a = 1 \implies a > 1$.
Solution: $a = 1 \wedge a \leq 1$.
Note: $\neg (P \implies Q) \equiv P \wedge \neg Q$.
Note: You should be able to write the negation of any statement, independently of thinking that it is true or, as above, false.

b) $b^2 \neq 4 \iff b \notin \{-2, 2\}$.
Solution: $b^2 \neq 4 \iff b \in \{-2, 2\}$ or $b^2 = 4 \iff b \notin \{-2, 2\}$.
Note: $\neg (P \iff Q) \equiv \neg P \iff Q \equiv P \iff \neg Q$.

c) $\forall x,\ x \geq 2 \implies -2x \geq 4$.
Solution: $\exists x,\ x \geq 2 \wedge -2x < 4$

Hint: Use tables 6 and 7 of [1], p. 24.

Exercise 6. For all natural number n, let x_n be a real number. Let $x_1 = 2$ and, for all $n \geq 1$, define

$$x_{n+1} = \frac{1}{2} x_n.$$

a) Compute x_2, x_3, x_4 and x_5.
Solution: $x_2 = \frac{1}{2} x_1 = 1$, $x_3 = \frac{1}{2}$, $x_4 = \frac{1}{4}$ and $x_5 = \frac{1}{8}$.

b) Show by induction that $\forall m \in \mathbb{N},\ x_m = 2^{2-m}$.
Solution: Let $P(n)$ be the statement $P(n) \stackrel{\Delta}{\equiv} x_n = 2^{2-m}$.
Basis step: $x_1 = 2 = 2^1 = 2^{2-1}$, so that $P(1)$ is true.
Induction step: Suppose $P(n)$ is true. Let's show that, in this case, $P(n+1)$ is also true. Since $P(n)$ is true, one has $x_n = 2^{2-n}$. Dividing each side of this equation by two, one gets $\frac{1}{2} x_n = 2^{2-n-1} = 2^{2-(n+1)}$. By definition of x_{n+1}, this last equation is $x_{n+1} = 2^{2-(n+1)}$, which is $P(n+1)$. See Section 3.3 of the textbook.

c) Let ε be a strictly positive real number. Show that there exists a natural number k such that $x_k < \varepsilon$.
Hint: $\varepsilon = 2^{\log_2 \varepsilon} > 2^{\lfloor \log_2 \varepsilon \rfloor - 1}$, where $\log_2$ is base-2 logarithm and $\lfloor \rfloor$ is the lower-rounding operation, so that $\lfloor x \rfloor$ is the largest integer not larger than x.
Solution: Let $k = 3 - \lfloor \log_2 \varepsilon \rfloor$. Then $x_k = 2^{2-k} = 2^{2-3+\lfloor \log_2 \varepsilon \rfloor} = 2^{-1+\lfloor \log_2 \varepsilon \rfloor} < 2^{\log_2 \varepsilon} = \varepsilon$.

Exercise 7. Suppose a person´s full name is composed of either

- a first name and a last name or
- a first name, a second name and a last name or
- a first name, a second name, a third name and a last name.

Moreover, there are 88799 different last names, and there are 4275 different female first names and 1219 male first names[1]. The second and third names of a person are taken from the same set as his/her first name. Let $\mathcal{L}$, $\mathcal{F}$ and $\mathcal{M}$ be the sets of last, female first and male first names, respectively.

a) Define the set full names, using the sets $\mathcal{L}$, $\mathcal{F}$, $\mathcal{M}$ and usual set operations such as union, intersection, complement, cross product etc.

[1] Source: U.S. Census Bureau `http://www.census.gov/genealogy/names/`.

Solution: The set of names of the form first-last, for both genders is $\mathcal{F} \times \mathcal{L} \cup \mathcal{M} \times \mathcal{L}$. The set of names of the form first-second-last is $\mathcal{F} \times \mathcal{F} \times \mathcal{L} \cup \mathcal{M} \times \mathcal{M} \times \mathcal{L}$. The set of names of the form first-second-third-last is $\mathcal{F} \times \mathcal{F} \times \mathcal{F} \times \mathcal{L} \cup \mathcal{M} \times \mathcal{M} \times \mathcal{M} \times \mathcal{L}$. Altogether, the set of full names is the union of these three sets:

$$\begin{aligned}&\mathcal{F} \times \mathcal{L} \cup \mathcal{M} \times \mathcal{L} \cup \\ &\mathcal{F} \times \mathcal{F} \times \mathcal{L} \cup \mathcal{M} \times \mathcal{M} \times \mathcal{L} \cup \\ &\mathcal{F} \times \mathcal{F} \times \mathcal{F} \times \mathcal{L} \cup \mathcal{M} \times \mathcal{M} \times \mathcal{M} \times \mathcal{L}.\end{aligned}$$

See Sec. 4.1 of the textbook.

b) How many distinct full names are there? Justify your answer.
Solution: Since the subsets in the union are disjoint, there are

$$\begin{aligned}|\mathcal{F} \times \mathcal{L}| + |\mathcal{M} \times \mathcal{L}| + |\mathcal{F} \times \mathcal{F} \times \mathcal{L}| + |\mathcal{M} \times \mathcal{M} \times \mathcal{L}| + \\ |\mathcal{F} \times \mathcal{F} \times \mathcal{F} \times \mathcal{L}| + |\mathcal{M} \times \mathcal{M} \times \mathcal{M} \times \mathcal{L}| &= \\ |\mathcal{F}| \cdot |\mathcal{L}| + |\mathcal{M}| \cdot |\mathcal{L}| + |\mathcal{F}|^2 \cdot |\mathcal{L}| + |\mathcal{M}|^2 \cdot |\mathcal{L}| + |\mathcal{F}|^3 \cdot |\mathcal{L}| + |\mathcal{M}|^3 \cdot |\mathcal{L}| &\simeq 7.1E15,\end{aligned}$$

Exercise 8. In a deck, there are 32 cards. Each card is of one of four possible suits and one of eight possible kinds, so that the deck can be identified with the set

$$\mathcal{D} = \{(s,k) \mid s \in \{1,2,3,4\}, k \in \{1,...,8\}\}.$$

A hand of eight cards is a set of eight distinct cards of the deck, i.e. the order of the card does not matter.

a) Define the set of hands of eight cards.
Solution: Using the set builder notation:

$$\begin{aligned}\{\{c_1,...,c_8\} \mid \forall i \in \{1,...,8\}, c_i \in \mathcal{D} \text{ and } \forall i \neq j, c_i \neq c_j\} &= \\ \{\{c_1,...,c_8\} \mid \forall i \in \{1,...,8\}, c_i \in \mathcal{D}\} &= \\ \{H \subseteq \mathcal{D} \mid |H| = 8\}&\end{aligned}$$

In the topmost expression, the part "$\forall i \neq j, c_i \neq c_j$" is not needed since "$\{c_1,...,c_8\}$" usually means that the c_i are all distinct.
See Sec. 1.6, especially Example 5 p. 79.

b) How many different hands of eight cards are there? That is, what is the cardinal of the set of hands of eight cards. Justify your answer.
Solution: This is the number of subsets of 8 out of 32 elements is

$$C(32,8) = 10518300$$

See Sec. 4.1.

c) How many different hands of eight cards are there in which all cards are of the same suit. Justify your answer.
Solution: 4.
Any single-suit hand of eight cards is defined uniquely by
1) Choosing a suit $s \in \{1,2,3,4\}$.
2) A subset of eight out of eight cards.
There are 4 possibilities at the first step and $C(8,8) = 1$ at the second step, so in all, $1 \cdot 4 = 4$ possibilities.

Example. Proof by induction.
Suppose we are given **the facts**

For all natural number n, let x_n be a real number. Let

$$x_1 = 1 \tag{8.1}$$

and, for all $n \geq 1$, define

$$x_{n+1} = x_n + \frac{1}{3}. \tag{8.2}$$

This means that

a) there is an infinite sequence of real numbers, where the n^{th} number is called x_n.
b) The first number of the sequence is 1.
c) I know that the $(n+1)^{\text{th}}$ number in the sequence is equal to the n^{th} number plus $\frac{1}{3}$.

Typically, we will want to compute the first few numbers in the sequence:

$$x_2 = x_1 + \frac{1}{3} = \frac{4}{3} \simeq 1.333,\ x_3 = x_2 + \frac{1}{3} = \frac{5}{3} \simeq 1.666,\ ...$$

If I am asked **the question**

Show by induction that

$$\forall m \geq 1,\ x_m = \frac{2+m}{3} \tag{8.3}$$

this means that I should show that

> for all natural number m greater or equal to 1, x_m is equal to $\frac{2+m}{3}$

or, put in still another way

> for all natural number m greater or equal to 1, the statement $x_m = \frac{2+m}{3}$ is true.

So, given the facts in Equations (8.1) and (8.2), I must be able to prove that, indeed, Equation (8.3) is true. Note that this is a property of all natural numbers.
How do I show that? In three steps:

a) Give a name, e.g. $P(m)$ to the statement that I want to prove for any number $m \geq 1$. In this case, $P(m)$ is the statement $x_m = \frac{2+m}{3}$.
b) "Basis step": I show that the statement $P(1)$ is true. That is[2], show that $x_1 = \frac{2+1}{3}$.
c) "Inductive step": I show that if, for some $k \geq 1$, the statement $P(k)$ is true[3], then the statement $P(k+1)$ is also true[4].

[2]Since $P(m) \equiv x_m = \frac{2+m}{3}$.
[3]That is, the statement $x_k = \frac{2+k}{3}$ is true.
[4]That is, $x_{k+1} = \frac{2+(k+1)}{3}$ is also true.

My answer will consist of

Define the statement $P(m)$ to be:

$$P(m) \stackrel{\Delta}{\equiv} x_m = \frac{2+m}{3}$$

In order to prove by induction that $P(m)$ is true for all $m \geq 1$, it is sufficient to show that:

Basis step. $P(1)$ is true. This is true because $P(1) \stackrel{\Delta}{\equiv} x_1 = \frac{2+1}{3}$ by definition of $P()$ and $x_1 = 1 = \frac{2+1}{3}$ by assumption[a].

Inductive step. Assume that, for some $k \geq 1$, $P(k)$ is true, i.e. one has

$$x_k = \frac{2+k}{3}.$$

One also has[b]

$$x_k + \frac{1}{3} = \frac{2+k}{3} + \frac{1}{3} = \frac{2+(k+1)}{3}.$$

Moreover, by definition[c] of x_{k+1} this expression is also equal to x_{k+1}, so that one has:

$$x_{k+1} = \frac{2+(k+1)}{3}.$$

This statement is $P(k+1)$, which is thus deduced from $P(k)$.

[a] The assumption mentioned here is Equation (8.1). The footnotes are not part of the answer to the question.
[b] By adding $\frac{1}{3}$ to both sides of this equation.
[c] This is Equation (8.2).

See also the examples of the textbook [1], pp. 240 onward.

References

[1] K. H. Rosen. *Discrete Mathematics and Its Applications.* Mc Graw Hill, 5 edition, 2003.