CS275 GRADED HOMEWORK

GIVE BACK ON TUESDAY SEP. 21ST 2004 AT BEGINNING OF CLASS

For each question, read **each word** with the greatest care and **without precipitation**. If you have doubts about what is asked, **go back** to the wording of the question until the meaning of the question is clear. Then try to find an answer. If you get stuck, don´t hesitate to **contact** your T.A. or me.

Please see also the example solution of a "prove by induction" exercise, at the end of this document.

Please write your section number on your homework as well as a rough estimate of the time you spent solving it.

Exercise 1. Let A, B and C be the sets defined by
$$
\begin{aligned}
A &= \left\{x \in \mathbb{Z} \mid x^2 < 9\right\} \\
B &= \left\{x \in \mathbb{N} \mid x^2 \leq 9\right\} \\
C &= \left\{x \in \mathbb{N} \mid x \leq 100 \wedge \exists y \in \mathbb{N}, y^2 = x\right\}
\end{aligned}
$$

   a) Determine the numbers of elements of A, B and C. That is, determine $|A|$, $|B|$ and $|C|$.

   b) Determine the number of elements in $A \cup B$, $A \cup C$, $A \cup C$, $A \cap B$, $A \cap C$, $B \cap C$.

   c) Determine the number of elements in $A \times B$, $A \times C$ and $A \times B \times C$.

Exercise 2. Write in English the following statement

$$\forall A \, \forall B, \, |A \cup B| = |A| + |B| \, .$$

Prove or disprove this statement.

Exercise 3. Write in English the following statement

$$\forall A \, \exists B, \, |A \cup B| \neq |A| + |B| \, .$$

Prove or disprove this statement.

Exercise 4. Write in English the following statement

$$\exists A \, \exists B, \, |A \times B| = |A| \cdot |B| \, .$$

Prove or disprove this statement.

Exercise 5. Write, in mathematical notation, the negations of the following statements

    a) $a = 1 \implies a > 1$.
    b) $b^2 \neq 4 \iff b \notin \{-2, 2\}$.
    c) $\forall x,\ x \geq 2 \implies -2x \geq 4$.

Hint: Use tables 6 and 7 of [1], p. 24.

Exercise 6. For all natural number n, let x_n be a real number. Let $x_1 = 2$ and, for all $n \geq 1$, define

$$x_{n+1} = \frac{1}{2}x_n.$$

    a) Compute x_2, x_3, x_4 and x_5.
    b) Show by induction that $\forall m \in \mathbb{N},\ x_m = 2^{2-m}$.
    c) Let ε be a strictly positive real number. Show that there exists a natural number k such that $x_k < \varepsilon$.
        Hint: $\varepsilon = 2^{\log_2 \varepsilon} < 2^{\lfloor \log_2 \varepsilon \rfloor - 1}$, where $\log_2$ is base-2 logarithm and $\lfloor \rfloor$ is the lower-rounding operation, so that $\lfloor x \rfloor$ is the largest integer not larger than x.

Exercise 7. Suppose a person´s full name is composed of either

    • a first name and a last name or
    • a first name, a second name and a last name or
    • a first name, a second name, a third name and a last name.

Moreover, there are 88799 different last names, and there are 4275 different female first names and 1219 male first names[1]. The second and third names of a person are taken from the same set as his/her first name. Let $\mathcal{L}$, $\mathcal{F}$ and $\mathcal{M}$ be the sets of last, female first and male first names, respectively.

    a) Define the set full names, using the sets $\mathcal{L}$, $\mathcal{F}$, $\mathcal{M}$ and usual set operations such as union, intersection, complement, cross product etc.
    b) How many distinct full names are there? Justify your answer.

Exercise 8. In a deck, there are 32 cards. Each card is of one of four possible suits and one of eight possible kinds, so that the deck can be identified with the set

$$\mathcal{D} = \{(s, k) \mid s \in \{1, 2, 3, 4\},\ k \in \{1, ..., 8\}\}.$$

A hand of eight cards is a set of eight distinct cards of the deck, i.e. the order of the card does not matter.

    a) Define the set of hands of eight cards using the set builder notation.
    b) How many different hands of eight cards are there? That is, what is the cardinal of the set of hands of eight cards. Justify your answer.
    c) How many different hands of eight cards are there in which all cards are of the same suit. Justify your answer.

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[1] Source: U.S. Census Bureau http://www.census.gov/genealogy/names/.

Example. Proof by induction.
Suppose we are given **the facts**

> For all natural number n, let x_n be a real number. Let
>
> (8.1) $$x_1 = 1$$
>
> and, for all $n \geq 1$, define
>
> (8.2) $$x_{n+1} = x_n + \frac{1}{3}.$$

This means that

- **a)** there is an infinite sequence of real numbers, where the nth number is called x_n.
- **b)** The first number of the sequence is 1.
- **c)** I know that the $(n+1)^{\text{th}}$ number in the sequence is equal to the nth number plus $\frac{1}{3}$.

Typically, we will want to compute the first few numbers in the sequence:

$$x_2 = x_1 + \frac{1}{3} = \frac{4}{3} \simeq 1.333, \ x_3 = x_2 + \frac{1}{3} = \frac{5}{3} \simeq 1.666, \ ...$$

If I am asked **the question**

> Show by induction that
>
> (8.3) $$\forall m \geq 1, \ x_m = \frac{2+m}{3}$$

this means that I should show that

> for all natural number m greater or equal to 1, x_m is equal to $\frac{2+m}{3}$

or, put in still another way

> for all natural number m greater or equal to 1, the statement $x_m = \frac{2+m}{3}$ is true.

So, given the facts in Equations (8.1) and (8.2), I must be able to prove that, indeed, Equation (8.3) is true. Note that this is a property of all natural numbers.
How do I show that? In three steps:

- **a)** Give a name, e.g. $P(m)$ to the statement that I want to prove for any number $m \geq 1$. In this case, $P(m)$ is the statement $x_m = \frac{2+m}{3}$.
- **b)** "Basis step": I show that the statement $P(1)$ is true. That is[2], show that $x_1 = \frac{2+1}{3}$.
- **c)** "Inductive step": I show that if, for some $k \geq 1$, the statement $P(k)$ is true[3], then the statement $P(k+1)$ is also true[4].

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[2]Since $P(m) \equiv x_m = \frac{2+m}{3}$.

[3]That is, the statement $x_k = \frac{2+k}{3}$ is true.

[4]That is, $x_{k+1} = \frac{2+(k+1)}{3}$ is also true.

My answer will consist of

Define the statement $P(m)$ to be:

$$P(m) \stackrel{\Delta}{\equiv} x_m = \frac{2+m}{3}$$

In order to prove by induction that $P(m)$ is true for all $m \geq 1$, it is sufficient to show that:

Basis step. $P(1)$ is true. This is true because $P(1) \stackrel{\Delta}{\equiv} x_1 = \frac{2+1}{3}$ by definition of $P()$ and $x_1 = 1 = \frac{2+1}{3}$ by assumption[a].

Inductive step. Assume that, for some $k \geq 1$, $P(k)$ is true, i.e. one has

$$x_k = \frac{2+k}{3}.$$

One also has[b]

$$x_k + \frac{1}{3} = \frac{2+k}{3} + \frac{1}{3} = \frac{2+(k+1)}{3}.$$

Moreover, by definition[c] of x_{k+1} this expression is also equal to x_{k+1}, so that one has:

$$x_{k+1} = \frac{2+(k+1)}{3}.$$

This statement is $P(k+1)$, which is thus deduced from $P(k)$.

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[a]The assumption mentionned here is Equation (8.1). The footnotes are not part of the answer to the question.
[b]By adding $\frac{1}{3}$ to both sides of this equation.
[c]This is Equation (8.2).

See also the examples of the textbook [1], pp. 240 onward.

REFERENCES

[1] K. H. Rosen. *Discrete Mathematics and Its Applications*. Mc Graw Hill, 5 edition, 2003.