DISCRETE MATH - CS275 FINAL EXAM

DECEMBER 14TH 2004, 1PM - 3PM

Each of the 4 exercises is worth 5 points.
No calculator is needed or allowed in this exam.
Justify your answers.

Advice:

- Read all the exercises before answering to any.
- Answering the wrong question does not help. Read each exercise with great care and **without hurrying**. If needed, read it **many times**, until the meaning of the questions is clear.
- Check for extra information on the blackboard.

Exercise 1. For each of the pairs of simple graphs in Figure 1.1, prove that the two graphs are isomorphic by displaying a graph isomorphism, or explain why they are not isomorphic.

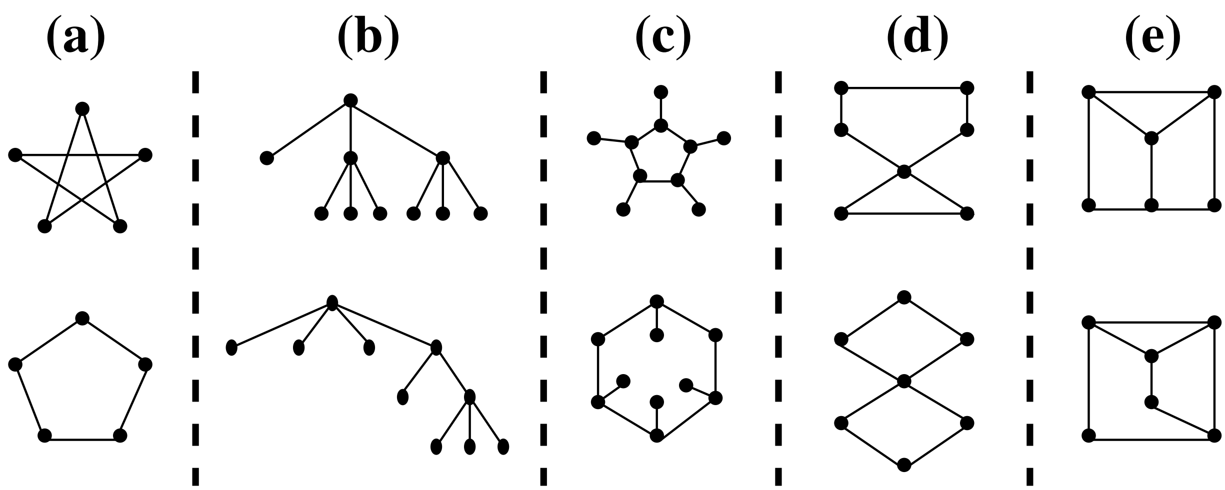

FIGURE 1.1. Graphs for Exercise 1.

Exercise 2. A six-sided dice is rolled 10 times. The outcome is represented by a sequence $X = (x_1, \dots, x_{10})$, where $x_i \in \{1, \dots, 6\}$ is the result of the i^{th} roll.

a) How many different outcomes are possible? Justify your answer.
b) What are the least and greatest possible values of $\sum_{i=1} x_i$? Justify your answer.
c) Write in mathematical notation the set of outcomes in which exactly one six comes out.
d) How many different outcomes are there in which exactly one six comes out? Justify your answer.
e) How many different outcomes are there in which $\sum_{i=1}^{10} x_i = 11$? Justify your answer.

Exercise 3. Let f and g be functions with domain $\mathbb{N}$ and co-domain $\mathbb{N}$, defined by:

$$\begin{aligned} f(x) &= 3\left\lceil \frac{x}{3} \right\rceil \quad \text{and} \\ g(x) &= x - 3\left\lceil \frac{x}{3} \right\rceil . \end{aligned}$$

Consider the following statements, which may be true or false:

A: $\forall y \in \mathbb{N}, \exists x \in \mathbb{N}, f(x) = y$.
B: $\forall y \in \mathbb{N}, \exists! x \in \mathbb{Z}, g(f(x)) = y$.
C: $\exists y \in \mathbb{N}, \forall x \in \mathbb{Z}, g(x) + f(x) \neq y$.

Answer the questions:

a) Write in English the statements **A**, **B** and **C**. If you like, you may use the words "one-to-one", "onto", "bijection" etc.
b) Prove or disprove each of the statements **A**, **B** and **C**.

Hint: You may find useful to compute by hand a few values of $f(x)$ and $g(x)$.

Exercise 4. We consider strings (sequences) of characters "A", "B" and "C", in which consecutive consonants are not allowed; that is, substrings "BB", "BC", "CB" and "CC" are not allowed.
Let x_n be the number of allowed strings of length n that end in "B" or "C" and let y_n be the number of allowed strings of length n that end in "A".

a) Write all the allowed strings of length one, two and three and write x_1, x_2, x_3, y_1, y_2 and y_3.
b) Notice that, for $n > 1$, all strings of length n ending in a consonant are obtained by appending a consonant to a message of length $n-1$ that ends in "A".
Write x_n as a function of y_{n-1}.
c) Notice that, for $n > 1$, all strings of length n ending in "A" are obtained by appending an "A" to a message of length $n-1$.
Write y_n as a function of y_{n-1} and x_{n-1}.
d) Using your answers to the previous questions, find a homogeneous recurrence relation verified by y_n. Solve this relation and write y_n as a function of n.
e) Write the total number of allowed strings, $x_n + y_n$ as a function of n only.